Lecture 21: Weak Solutions

- Roughly speaking, weak solutions are non-differentiable solutions that "satisfy" the PDE in an appropriate sense

- They arise due to perhaps non-differentiable initial data where a concept of a solution may still be important.

Test Functions

- We call $C_c^\infty(\mathcal{U})$ the space of test functions on $\mathcal{U}$, with the idea that we "test against them".

- Indeed, let $u \in C^0(\mathcal{U})$. Then, if $\int_{\mathcal{U}} u \cdot \varphi \, dx = 0$ for all $\varphi \in C_c^\infty(\mathcal{U})$, we must have $u = 0$.

Pf Assume for contradiction $u(x_0) > 0$. Then, by continuity, there exists some $B(x_0, \delta)$ on which

$u(y) > \frac{1}{2} u(x_0) > 0$. Let $\varphi(x)$ be a smooth positive bump supported in $B(x_0, \delta)$ and $\varphi(x_0) = 1$. Then,

$\int_{\mathcal{U}} u \varphi \, dx > 0$. ☒

In concept, C_c^∞ lets us "probe" functions.

- Further, if $u \in C^1(\mathbb{R})$, we may "detect" the derivative via $\varphi \in C_c^\infty$: For all $\varphi \in C_c^\infty(\mathcal{U})$ (and $\mathcal{U} \in \mathbb{R}$).

$$\int_{\mathcal{U}} u' \varphi \, dx = \int_{\mathcal{U}} u(-\varphi') \, dx$$

where the RHS "works" even if $u \notin C^1$.

- We use this concept to define a weak derivative.

$u' = f$ if $\int_{\mathcal{U}} -u \varphi' \, dx = \int_{\mathcal{U}} f \varphi \, dx \quad \forall \varphi \in C_c^\infty(\mathcal{U})$.

• With that in mind, we consider the most general ambient space where these ideas make sense:

$$\mathcal{L}^1_{loc}(U) = \{ f : U \to \mathbb{C} \; ; \; f|_K \in \mathcal{L}^1(K) \text{ for all compact } K \subset U \}$$

$\boxed{\text{Lemma: } 10.1}$ If $f \in \mathcal{L}^1_{loc}(U)$ satisfies $\int_U f \varphi \, dx = 0$ for all $\varphi \in C_c^\infty(U)$, then $f \equiv 0$.

$\boxed{\text{Pf}}$ Consider any $K \subseteq \Omega$ compact. Then, there exists some $r > 0$ such that for all $0 < \varepsilon < r$, the "bubble" $B(K, \varepsilon) \subseteq \Omega$. We create a smooth φ_ε so $\varphi_\varepsilon \equiv 1$ on K and $\mathrm{supp}(\varphi_\varepsilon) \subset B(K, \varepsilon)$. Then, $f \cdot \varphi_\varepsilon = f_\varepsilon \in \mathcal{L}^2(\Omega)$. Pick $\{\varphi_k\} \subset C_c^\infty(\Omega)$ so $\varphi_k \to f_\varepsilon$ in $\mathcal{L}^2$. Notice that $\int f_\varepsilon \varphi_k \, dx = \int f \varphi_\varepsilon \varphi_k \, dx = 0$, so $\|f_\varepsilon\|_2 = 0$. Hence, $f_\varepsilon \equiv 0$ ($f_\varepsilon = 0$ a.e.).

We then must have $f \equiv 0$. $\square$

ex.) In $\mathbb{R}$, consider $g(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 0 \end{cases}$. We expect $g'(x)$ to look like $f(x) = \begin{cases} 0 & x \notin [0,1] \\ 1 & x \in (0,1) \end{cases}$.

In terms of weak derivatives,

$$\int_{-\infty}^{\infty} g \varphi' \, dx = \int_1^\infty \varphi' \, dx + \int_0^1 x \varphi' \, dx$$
$$= -\varphi(1) + [x \varphi|_0^1] - \int_0^1 \varphi \, dx$$
$$= -\int_0^1 \varphi \, dx = -\int_{\mathbb{R}} f \varphi \, dx$$

So $g' = f$ weakly.

ex.) For $t \in \mathbb{R}$, define $w \in \mathcal{L}^1_{loc}(U)$ by
$w(t) = \begin{cases} w_-(t) & t < 0 \\ w_+(t) & t \geq 0 \end{cases}$ for $w_-, w_+ \in C^1(\mathbb{R})$.

For $\varphi \in C_c^\infty(\mathbb{R})$, $-\int_{-\infty}^{\infty} w(t) \varphi'(t) \, dt = \int_{-\infty}^0 -w_- \varphi' \, dt + \int_0^\infty -w_+ \varphi' \, dt$
$= [w_+(0) - w_-(0)] + \int_{-\infty}^{\infty} w_+' \varphi \, dt + \int_{-\infty}^0 w_-' \varphi \, dt$

So if $w_+(0) = w_-(0)$, $w'(t) = \begin{cases} w_-'(t) & t < 0 \\ w_+'(t) & t > 0 \end{cases}$ weakly.

ex.) Consider $H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$. For $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} -H(t)\varphi'(t)\,dt = \int_0^\infty -\varphi'(t)\,dt = \varphi(0)$$

So we consider $H'(t)$ to be the "point mass" or evaluation $\delta_0(x)$ so $\int \delta_0(x)\varphi(x)\,dx = \varphi(0)$.

• Multi-Indices:

- we introduce a notation to simplify writing partials.

For each multi-index $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n)$ with $\alpha_j \in \mathbb{N}_0$, we denote

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

with order $|\alpha| = \alpha_1 + \dots + \alpha_n$

e.g. if $u, \varphi \in C_c^\infty(\Omega)$, $\int_\Omega (D^\alpha u)\varphi\,dx = (-1)^{|\alpha|}\int_\Omega u D^\alpha \varphi\,dx.$

$\boxed{\text{Th}^{\underline{m}}\ 10.4}$ If $u \in C^m(U)$, then u is weakly differentiable to order m and the weak and classical derivatives coincide.

Conversely, if $u \in L^1_{loc}(U)$ has ~~class~~ weak derivatives $D^\alpha u$ for $|\alpha| \le m$ and each $D^\alpha u$ is continuous (or equivalent to a continuous function), then u is equivalent to a $C^m(U)$ function.

$\boxed{\text{Pf}}$ The first direction is integration-by-parts & lemma 10.1. The second direction relies on appropriate convergences we haven't built up analysis for (e.g. mollification). $\square$

Weak Solutions of Continuity Equations

•) Consider $\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$ $\quad u|_{t=0} = g$ for $u(t,x)$, with flux $q(t,x)$ as we derived for the method-of-characteristics.

Suppose u is a classical solution & q is differentiable.

Let $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$. Then,

$$\int_0^\infty \frac{\partial u}{\partial t}\varphi\,dt = -u\varphi|_{t=0} - \int_0^\infty u\frac{\partial \varphi}{\partial t}\,dt$$

but

$$\int_0^\infty \frac{\partial q}{\partial x}\varphi\,dt = -\int_0^\infty q\frac{\partial \varphi}{\partial x}\,dx$$

such that

$$\int_0^\infty\int_{-\infty}^\infty \left[\partial u/\partial t + \partial q/\partial x\right]\varphi\,dx\,dt = -\int_0^\infty\int_{-\infty}^\infty u\frac{\partial \varphi}{\partial t} + q\frac{\partial \varphi}{\partial x}\,dx\,dt - \int_{-\infty}^\infty u\varphi|_{t=0}\,dx$$

•) If u is a classical solution,

$$\int_0^\infty\int_{-\infty}^\infty u\frac{\partial \varphi}{\partial t} + q\frac{\partial \varphi}{\partial x}\,dx\,dt + \int_{-\infty}^\infty g\,\varphi|_{t=0}\,dx = 0 \qquad (A)$$

for all $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$. For g, q locally integrable, we define $u(t,x)$ to be a weak solution to

$$\begin{cases} \partial u/\partial t + \partial q/\partial x = 0 \\ u|_{t=0} = g \end{cases}$$

if (A) holds for all $\varphi \in C_c^\infty$.

ex.) Consider $q(t,x) = c\cdot u(t,x)$ $\quad (c \in \mathbb{R})$. By the method of characteristics, $u(t,x) = g(x-ct)$.

If $g \in L^1_{loc}(\mathbb{R})$, this defines a weak solution:

$$\int_0^\infty\int_{-\infty}^\infty g(x-ct)\left[\frac{\partial \varphi}{\partial t}(t,x) - c\frac{\partial \varphi}{\partial x}(t,x)\right]dx\,dt$$

is for $\tau = t$, $y = x - ct$

$$\int_0^\infty\int_{-\infty}^\infty g(y)\frac{\partial \tilde\varphi}{\partial \tau}(\tau,y)\,dy\,d\tau \quad \text{for } \tilde\varphi(\tau,y) = \varphi(\tau, y+c\tau).$$

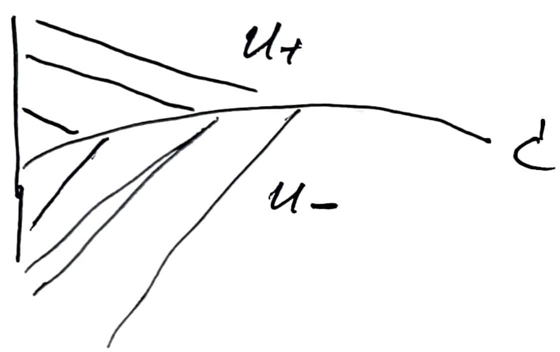
$$= \int_{-\infty}^\infty g(y)\left(-\tilde\varphi(0,y)\right)dy$$

$$= \int_{-\infty}^\infty -g(y)\varphi(0,y)\,dy \qquad \text{as desired.}$$

Rmk: Previously, we noted that jump discontinuities are a bit beyond standard weak derivatives here. The above works because we see regularity along characteristics.

- Let us consider another equation $\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} q(u) = 0$ (B) for $q: \mathbb{R} \to \mathbb{R}$ smooth. The characteristics are straight lines whose slope depends on initial conditions.

  - In such cases, we saw shocks form as characteristics crossed. One possible way to resolve this is to draw a shock curve C and pick classical solutions above and below the line, then give some "jump condition"



$\boxed{\text{Thm } 10.6}$ Rankine-Hugoniot Condition

Let C be characterized by $x = G(t)$ with $G \in C^1([0,\infty))$. Suppose u is a weak solution of (B) given by

$$u(t,x) = \begin{cases} u_-(t,x) & x < G(t) \\ u_+(t,x) & x > G(t) \end{cases}$$

where u_+, u_- are classical solutions. Then, along C,

$$q(u_+) - q(u_-) = (u_+ - u_-)G'$$

$\boxed{\text{Pf}}$ Consider $\psi \in C_c^\infty((0,\infty)\times\mathbb{R})$ so that by the weak solution defn, (b/c $(0,\infty)\times\mathbb{R}$ open)

$$\int_0^\infty \int_{-\infty}^\infty \left[u \frac{\partial \psi}{\partial t} + q(u) \frac{\partial \psi}{\partial x} \right] dx\,dt = 0$$

By our assumptions, this is

$$\int_0^\infty \int_{-\infty}^{G(t)} q(u_-) \frac{\partial \psi}{\partial x} dx\,dt + \int_0^\infty \int_{G(t)}^\infty q(u_+) \frac{\partial \psi}{\partial x} dx\,dt + \int_0^\infty \int_{-\infty}^{G(t)} u_- \frac{\partial \psi}{\partial t} dx\,dt$$

$$+ \int_0^\infty \int_{G(t)}^\infty u_+ \frac{\partial \psi}{\partial t} dx\,dt$$

•) Set $A_1 = \{(t,x) \in (0,\infty)\times\mathbb{R} : x < \sigma(t)\}$.

Let $F = \langle u, q(u)\rangle$ and the above has term

$$\iint_{A_1} F\cdot\nabla_{t,x}\varphi\,dx\,dt = -\iint_{A_1}\varphi\,\mathrm{div}(F)\,dx\,dt + \int_{\partial A_1}\varphi\,\eta\cdot F\,dS$$

$$= -\iint_{A_1}\varphi\left[\frac{\partial u_-}{\partial t} - \frac{\partial}{\partial x}q(u_-)\right]dx\,dt + \int_0^\infty \varphi\langle u, q(u)\rangle\cdot\langle -\sigma'(t), 1\rangle\,dt$$

$\to 0$ by assumption

$$= -\iint_{A_1}\varphi\left[\partial u_-/\partial t - \frac{\partial}{\partial x}q(u_-)\right]dx\,dt + \int_0^\infty q(u_-)\varphi - \varphi u_-\sigma'\,dt\Big|_{x=\sigma(t)}$$

Repeating on $A_2 = \{(t,x)\in(0,\infty)\times\mathbb{R} : x > \sigma(t)\}$ gives

$$\int_0^\infty\int_{-\infty}^\infty u\frac{\partial\varphi}{\partial t} + q(u)\frac{\partial\varphi}{\partial x}\,dx\,dt = \int_0^\infty \left[(u_+ - u_-)\varphi\sigma' - (q(u_+) - q(u_-))\varphi\right]_{x=\sigma(t)}dt$$

such that $\int_0^\infty \varphi\left[(u_+ - u_-)\sigma' - (q(u_+) - q(u_-))\right]_{x=\sigma(t)}dt = 0$

for all such φ, or

$(u_+ - u_-)\sigma' - (q(u_+) - q(u_-)) = 0$, on $x = \sigma(t)$. $\square$

ex.) Consider the traffic equation $\partial u/\partial t + (1-2u)\,\partial u/\partial x = 0$

$(q(u) = u - u^2)$ with

$$u(0,x) = \begin{cases} a & x<0 \\ b & x>0\end{cases}$$

Characteristics are

$$x(t) = \begin{cases} x_0 + (1-2a)t & x_0 < 0 \\ x_0 + (1-2b)t & x_0 > 0\end{cases}$$

If $a<b$, these give a shock

The solutions above & below the shock lines are constant:

$u_- = a$, $u_+ = b$.

Thus, the R-H condition is $(b-b^2) - (a-a^2) = (b-a)\sigma'$

So $\sigma' = (1-b-a)$

or

$$u(t,x) = \begin{cases} a & x < (1-b-a)t \\ b & x > (1-b-a)t \end{cases}$$

(Shock starts at origin)